THE GREEN CONJECTURE FOR EXCEPTIONAL CURVES ON A $K3$ SURFACE

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Abstract. We use the Brill-Noether theory to prove the Green conjecture for exceptional curves on $K3$ surfaces. Such curves count among the few ones having Clifford dimension ≥ 3. We obtain our result by adopting an infinitesimal approach due to Pareschi, and using the degenerate version of the Hirschowitz-Ramanan-Voisin theorem obtained in [A05].

1. Introduction.

Two conjectures made in the eighties by Green, and Green-Lazarsfeld, pointed out to some deep links between the intrinsic properties of algebraic curves, and their Koszul cohomology groups with values in suitably chosen line bundles. Recall that the Koszul cohomology $K_{p,q}(X,L)$ of a complex projective variety X with values in a globally generated line bundle L on X is defined as the cohomology at the middle of the complex:

$$\bigwedge^{p+1} H^0(L)\otimes H^0(L^{\otimes(q-1)}) \to \bigwedge^{p} H^0(L)\otimes H^0(L^{\otimes q}) \to \bigwedge^{p-1} H^0(L)\otimes H^0(L^{\otimes(q+1)}).$$

A basic result due to Green and Lazarsfeld, see [G84, Appendix], shows that $K_{r_1+r_2-1,1}(X,L)$ is not zero if X is smooth and L is decomposed as $L = L_1 + L_2$ with $r_i := h^0(X,L_i) - 1 \geq 1$. In the particular case of curves, one obtains that $K_{g-c-2,1}(X,K_X)$ and $K_{h^0(L)-d-1,1}(X,L)$ are not zero, where g is the genus of X, c is the Clifford index (see [Ma82] for the definition), d is the gonality, and L is an arbitrary line bundle of sufficiently large degree. The above-quoted conjectures predict that these non-vanishing results are sharp.

Conjecture 1.1 (Green, [G84]). $K_{g-c-1,1}(X,K_X) = 0$

Conjecture 1.2 (Green-Lazarsfeld, [GL85]). $K_{h^0(L)-d,1}(X,L) = 0$

In recent years, strong evidence has been found for these conjectures. Putting together Voisin's [V02] and Teixidor's [T02] results, Green conjecture is valid for a generic d-gonal curve, for $d < [g/2] + 2$. The case of generic curves of odd genus, not covered by the above-mentioned results was settled by Voisin [V05]. This case was particularly challenging due to a previous work of Hirschowitz and Ramanan [HR98], which together with Voisin's result [V05] implies the validity of the Green conjecture for *any* curve of odd genus and maximal gonality. The Green-Lazarsfeld conjecture is also valid for generic d-gonal curves, see [AV03], [A04], and [A05].

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The Brill-Noether theory could play a major rôle in the attempt to solve these conjectures. Specifically, suppose $d \geq 3$ is an integer, and X is a smooth d-gonal curve with $d < [g/2] + 2$, such that the varieties $W^1_{d+n}(X)$, parametrising degree-$(d+n)$ pencils on X, verify

(Linear growth conditions): $\dim(W^1_{d+n}(X)) \leq n$, for all n such that $0 \leq n \leq g - 2d + 2$.

Then X verifies both Green, and Green-Lazarsfeld conjectures, see [A05, Theorem 2].

It is therefore important to try and control the dimensions of the varieties of pencils, and find intervals on which their growth is linear in the degree. A prediction was made by G. Martens, [Ma84, Statement (T)]:

Statement (T). *If $j \geq 1$, $g \geq 2j + 4$, and $m \in [j + 3, g - 1 - j]$ are integer numbers, and X is a curve of genus g, such that* $\dim(W^1_m(X)) = m - 2 - j$*, then* $\dim(W^1_s(X)) = s - 2 - j$ *for all integers $s \in [j + 3, g - j]$.*

Notice that one of the conditions above is $\dim(W^1_{j+3}(X)) = 1$, and the inequality $d \geq j + 3$ yields to $j + 3 \leq (g + 1)/2$. In particular, for Brill-Noether generic curves, the Statement (T) is empty. For explicit special curves of Clifford dimension one, Statement (T) with $m = \mathrm{gon}(X) + 1$, together with [A05, Theorem 2] can be seen as a potential tool for verifying the Green, and Green-Lazarsfeld conjectures for new classes of curves.

Partial versions of Statement (T) have been proved (see for instance [Ho82], [Ma84], [CKM92]). Unfortunately, the picture happens to be more subtle than expected (and hoped). We exhibit counter-examples to Statement (T) as special curves in the linear system $|2H|$ on a $K3$ surface whose Picard group is generated by a hyperplane section H and a line ℓ.

Proposition 1.3. *Let S be a $K3$ surface with* $\mathrm{Pic}(S) = \mathbb{Z}.H \oplus \mathbb{Z}\ell$*, with H very ample, $H^2 = 2r - 2$, and $H.\ell = 1$. There exists a smooth curve $C \in |2H|$ (whose genus equals $4r - 3$, and whose gonality equals $2r - 2$, see §3.1), and*

(1) $\dim(W^1_{2r-2}(C)) = 1$.
(2) $\dim(W^1_{2r-1}(C)) = 2$.
(3) $\dim(W^1_{2r}(C)) = 4$.
(4) $\dim(W^1_{2r+1}(C))$ *equals* 5 *or* 6.

Nevertheless, we prove that the generic curves in $|2H|$ verify a weak linear growth condition.

Theorem 1.4. *Let S be a $K3$ surface with* $\mathrm{Pic}(S) = \mathbb{Z}.H \oplus \mathbb{Z}\ell$*, with H very ample, $H^2 = 2r - 2 \geq 4$, and $H.\ell = 1$. Then the generic curve in the linear system $|2H|$ verifies*

$$\dim(W^1_{2r-2+n}(C)) = n, \quad \textit{for } n \in \{0, 1, 2\}.$$

A consequence of this result is that the Green-Lazarsfeld conjecture holds for the generic curves in the linear system $|2H|$, see Corollary 4.5.

The idea of the proof is to look at the family of pairs (C, A), with $C \in |2H|$ smooth and $A \in W^1_{2r-2+n}(C)$, and to give a bound on the dimension of the irreducible

components dominating $|2H|$. Thanks to the work of Lazarsfeld and Mukai, to the data (C, A) (for simplicity we assume here that A is a complete and base-point-free pencil) one can attach a rank 2 vector bundle $E(C, A)$ on the surface S. If this bundle is *simple*, then the original argument of Lazarsfeld's [L], or the variant provided by Pareschi [P95], allows one to determine these dimensions. In the *non-simple* case a useful lemma (see [GL87], [DM89] and [CP95]), brings to a very concrete description of the parameter space for such bundles. This description, together with the infinitesimal approach of Pareschi [P95], allows us to conclude.

Moreover, and more importantly, using the degenerate version of the Hirschowitz-Ramanan-Voisin theorem [A05, Proposition 8], we derive from Theorem 1.4 the following.

Theorem 1.5. *Let S be a $K3$ surface with $\mathrm{Pic}(S) = \mathbb{Z}.H \oplus \mathbb{Z}\ell$, with H very ample, $H^2 = 2r - 2 \geq 4$, and $H.\ell = 1$. Then any smooth curve in the linear system $|2H + \ell|$ verifies the Green conjecture.*

Smooth curves in the linear system $|2H + \ell|$ are particularly interesting in several regards. First of all, they count among the few examples of curves whose Clifford index is not computed by pencils, i.e. $\mathrm{Cliff}(C) = \mathrm{gon}(C) - 3$, as it was shown in [ELMS89] (other obvious examples are given by plane curves, for which the Green conjecture is already checked, see [Lo89]). Such curves are the most special in the moduli space of curves from the Clifford index view-point, reason for which some authors call them *exceptional* curves. Hence, the case of smooth curves in $|2H + \ell|$ may be considered as opposite to that of generic curve of fixed gonality. Secondly, they carry a one-parameter family of pencils of minimal degree (see [ELMS89]), so [A05, Theorem 2] cannot be applied directly.

The outline of the article is the following. First, we recall some vector bundle techniques which will be used in the proof of Theorems 1.4 and 1.5. The proof of Theorem 1.4 splits in several cases. In §3.1 we treat the case $n = 0, 1$. In §3.2, we analyse the case $n = 2$, and $E(C, A)$ simple. The remaining case, $n = 2$, and $E(C, A)$ not simple are ruled out in §3.3. We put together all these intermediate steps in §3.4 and prove Theorem 1.4 and Proposition 1.3. Finally, we show how Theorem 1.4 implies Theorem 1.5.

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2. Vector bundle techniques.

In the present Section we recall some basic vector bundle techniques, see [La86], [GL87], [La89], [OSS80].

2.1. **The Lazarsfeld-Mukai vector bundle.** Given a smooth curve C belonging to a linear system $|L|$ on a $K3$ surface S and a base-point-free line bundle $A \in \mathrm{Pic}(C)$, we recall how to associate to this data, following [La86] and [Mu89], a vector bundle $E := E(C, A)$ of rank $h^0(C, A)$ on S and we record a number of properties of E which will be freely used in the rest of the paper. First one considers the rank

$h^0(C,A)$ vector bundle $F(C,A)$ defined as the kernel of the evaluation of sections of A (considered as torsion sheaf on the whole surface S)

$$0 \to F(C,A) \to H^0(C,A) \otimes \mathcal{O}_S \xrightarrow{ev} A \to 0. \tag{2.1.1}$$

Then dualizing the above exact sequence and setting $E := E(C,A) := F(C,A)^*$ we get :

$$0 \to H^0(C,A)^* \otimes \mathcal{O}_S \to E \to K_C(-A) \to 0. \tag{2.1.2}$$

The invariants of E are :

(1) $\det(E) = \mathcal{O}_S(C)$;
(2) $c_2(E) = \deg(A)$;
(3) $h^0(S,E) = h^0(C,A) + h^1(C,A) = 2h^0(C,A) - \deg(A) - 1 + g(C)$.

Moreover, E is globally generated off a finite set, and

$$h^1(S,E) = h^2(S,E) = 0.$$

There is a natural rational map

$$d_E : \mathrm{Gr}(\mathrm{rk}(E), H^0(S,E)) \dashrightarrow |L| \tag{2.1.3}$$

from the Grassmanniann of $\mathrm{rk}(E)$-dimensional subspaces of $H^0(S,E)$ to the linear system $|L|$. This map sends a generic subspace $\Lambda \in \mathrm{Gr}(\mathrm{rk}(E), H^0(S,E))$ to the degeneracy locus of the evaluation map :

$$\mathrm{ev}_\Lambda : \Lambda \otimes \mathcal{O}_S \to E$$

(notice that, generically, this degeneracy locus cannot be the whole surface, since E is generated off a finite set).

Moreover, for generic Λ, the image $d_E(\Lambda)$ is a smooth curve C_Λ (the smoothness of the degeneracy locus is an open condition, and it is realized on a non-empty set of $\mathrm{Gr}(\mathrm{rk}(E), H^0(S,E))$ by the construction of E), and the cokernel of ev_Λ is a line bundle $K_{C_\Lambda}(-A_\Lambda)$ of C_Λ, where $\deg(A_\Lambda) = c_2(E)$. An important feature on the map d_E is that its differential at a point Λ coincides with the multiplication map

$$\mu_{0,A_\Lambda} : H^0(C_\Lambda, A_\Lambda) \otimes H^0(C_\Lambda, K_{C_\Lambda} - A_\Lambda) \to H^0(C_\Lambda, K_{C_\Lambda}).$$

On the other hand, if M_A is the vector bundle (of rank $h^0(C,A)-1$) on C defined by the kernel of the evaluation map *on the curve* :

$$0 \to M_A \to H^0(C,A) \otimes \mathcal{O}_C \xrightarrow{ev} A \to 0, \tag{2.1.4}$$

by tensoring (2.1.4) with $K_C(-A)$, one gets

$$\ker(\mu_{0,A}) = H^0(C, M_A \otimes K_C(-A)). \tag{2.1.5}$$

Notice that, by construction, there is a natural surjective map from $F(C,A)_{|C}$ to M_A, and, by determinant reason, we have

$$0 \to A(-K_C) \to F(C,A)_{|C} \to M_A \to 0. \tag{2.1.6}$$

2.2. **The Serre construction.** The Serre construction consists in associating to a locally complete intersection 0-dimensional subscheme ξ of S, and to a non-zero element $t \in H^1(S, L \otimes I_\xi)^*$, where $L \in \mathrm{Pic}(S)$, a rank two vector bundle $E := E_{\xi,t}$ on S and a global section $s \in H^0(S, E)$, whose zero locus is ξ.

By the Grothendieck-Serre duality Theorem, we have

$$H^1(S, L \otimes I_\xi)^* \cong \mathrm{Ext}^1(L \otimes I_\xi, \mathcal{O}_S),$$

hence to each $t \in H^1(S, L \otimes I_\xi)^*$ we may associate a sheaf $\mathcal{E}$ is given by an extension

$$0 \to \mathcal{O}_S \to \mathcal{E} \to L \otimes I_\xi \to 0.$$

Notice that the global section s of $\mathcal{E}$ coming from the inclusion $\mathcal{O}_S \hookrightarrow \mathcal{E}$ vanishes on ξ.

The precise criterion for an extension as above to be locally free is the following:

Proposition 2.1 ([OSS80], [La97])**.** *Given $t \in H^1(S, L \otimes I_\xi)^*$, the corresponding $\mathcal{E}$ fails to be locally free if and only if there exists a proper (possibly empty!) subscheme $\xi' \subset \xi$ such that*

$$t \in \mathrm{Im}(H^1(S, L \otimes I_{\xi'})^* \to H^1(S, L \otimes I_\xi)^*).$$

We also recall a result which we state and only when S is a $K3$ surface (note that the canonical bundle of S is trivial).

Theorem 2.2 ([GH78])**.** *There exists a rank two vector bundle E on S, with $\det(E) = L$, and a section $s \in H^0(S, E)$ such that $V(s) = \xi$ if, and only if, every section of L vanishing at all but one of the points in the support of ξ also vanishes at the remaining point.*

Theorem 2.2 immediately yields :

Proposition 2.3. *Let S be a $K3$ surface, and L a line bundle on S. Then, for any 0-dimensional subscheme ξ of S, such that $h^0(S, L \otimes I_{\xi'}) = 0$, for all $\xi' \subset \xi$ with $\mathrm{lg}(\xi') = \mathrm{lg}(\xi) - 1$, there exists a rank two vector bundle E on S given by an extension*

$$0 \to \mathcal{O}_S \to E \to L \otimes I_\xi \to 0.$$

2.3. **A useful lemma for non-simple E.** Our analysis of the dominating irreducible components of the variety of pairs (C, A), where C is a smooth curve in $|2H|$, A a pencil on C, and the Lazarsfeld-Mukai associated vector bundle $E(C, A)$ is not simple, is based on the following key lemma (see [GL87], [DM89]. See [CP95, Lemma 2.1] for this precise statement).

Lemma 2.4. *Let S be a $K3$ surface, C a smooth curve on S and A a base-point-free line bundle on C, such that $h^0(C, A) = 2$. If $E(C, A)$ is not a simple vector bundle, then there exists two line bundles M and N on S and a 0-dimensional subscheme ξ of S such that*

(1) $h^0(S, M) \geq 2$, $h^0(S, N) \geq 2$;
(2) *N is base-point-free;*
(3) *there is an exact sequence*

$$0 \to M \to E \to N \otimes I_\xi \to 0.$$

Moreover if $h^0(M - N) = 0$, then $\mathrm{supp}(\xi) = \emptyset$ and the above sequence is split.

This result allows to describe the parameter space for non-simple Lazarsfeld-Mukai vector bundles $E(C,A)$ as an open subset of a projective bundle over the Hilbert scheme of points on S. We shall use this lemma for $\lg(\xi)=2$ or $\xi=\emptyset$.

3. Pencils on curves in $|2H|$.

3.1. **The invariants of curves in $|2H|$.** Let C be a smooth curve in the linear system. The adjunction formula computes the genus $g(C)=4r-3$.

The other basic invariants, the Clifford index and the gonality are obtained as follows.

Observe that $H|_C$ contributes to the Clifford index of C, and

$$\mathrm{Cliff}(H|_C)=2r-4.$$

In particular, $\mathrm{Cliff}(C)\leq 2r-4$, whereas the Clifford index of generic curves of genus $4r-3$ equals $2r-2$. By the main result of [GL87], the Clifford index of C is computed by a line bundle $N=aH+b\ell$ on the surface S. Let $M=(2-a)H-b\ell$. The bundle N can be assumed to be base-point-free, and $h^1(N)=0$, and $h^0(N)=h^0(N|_C)$, and we may assume that $h^1(M)=0$, and the restriction map $H^0(M)\to H^0(M|_C)$ is surjective (compare to [CP95, Proof of Proposition 3.3]). Since both $N|_C$, and $M|_C$ compute the Clifford index of C, we obtain $a=1$. The other conditions imply $b=0$, hence

$$\mathrm{Cliff}(C)=2r-4.$$

Finally, apply [CP95, Proposition 3.3] to conclude the the gonality of C equals $2r-2$.

3.2. **The parameter space of pairs, and the proof strategy.** The main ingredient used in the proof of Theorem 1.4 is the parameter space of pairs (C,A), where $C\in|2H|$ is a smooth curve, and A is a pencil of given degree m on C. Denote $|2H|_s$ the open subset of $|2H|$ corresponding to smooth curves. Following [AC81], there exists a variety $\mathcal{W}^1_m(|2H|_s)$, and a projective morphism

$$\pi_S:\mathcal{W}^1_m(|2H|_s)\to|2H|_s,$$

whose fibre $\pi_S^{-1}[C]$ over any $C\in|2H|_s$ is scheme-theoretically isomorphic to the variety of special divisors $W^1_m(C)$. The proof idea of Theorem 1.4 is to estimate the relative dimension of the dominating irreducible components of $\mathcal{W}^1_{2r-2+n}(|2H|_s)$, for $n\in\{0,1,2\}$.

Notice that the case $n=0$ has already been settled in [CP95, Theorem 3.1, and Lemma 3.2]: a generic member of the linear system $|2H|$ carries only finitely many minimal pencils (remark that $\rho(4r-3,1,2r-2)=-3$).

The case $n=1$ will follow as an immediate consequence of [CP95, Theorem 3.1, and Lemma 3.2], and of a Lemma of Accola (see [Ac81], cf. [ELMS89, Lemma 3.1], and [CM91]).

Lemma 3.1. *There is no base-point-free $\mathfrak{g}^1_{2r-1}$ on a smooth $C\in|2H|$.*

Proof. Indeed, if A was such a pencil, then by Accola's Lemma, we would have (the bundle $\mathcal{O}_C(H)$ is semi-canonical)

$$2h^0(C,\mathcal{O}_C(H-A))\geq 2h^0(C,\mathcal{O}_C(H))-(2r-1).$$

Since $h^0(S, \mathcal{O}_S(H)) = h^0(C, \mathcal{O}_C(H)) = r+1$, we obtain $2h^0(C, \mathcal{O}_C(H-A)) \geq 3$, i.e. $h^0(C, \mathcal{O}_C(H-A)) \geq 2$. In this case, $\mathcal{O}_C(H-A)$ would be a pencil of degree $2r-3$, contradicting $\mathrm{gon}(C) = 2r-2$. □

Therefore, for smooth curves $C \in |2H|$, the variety $W^1_{2r-1}(C)$ is 1-dimensional, and its irreducible components are obtained by adding base-points to the (finitely many) $\mathfrak{g}^1_{2r-2}$'s on C.

So, we may suppose $n = 2$. Let $\mathcal{W}$ be an irreducible component of $\mathcal{W}^1_{2r}(|2H|_s)$, such that for a generic pair (C, A), the line bundle A is base-point-free. By [ACGH85, Lemma 3.5, p. 182], we necessarily have $h^0(C, A) = 2$. We distinguish two cases according to the behaviour of the associated Lazarsfeld-Mukai bundle $E(C, A)$. Precisely, components whose generic member (C, A) has *simple*, respectively *non simple*, Lazarsfeld-Mukai vector bundle are called *simple components*, respectively *non simple components*.

These two cases are treated in separate Subsections in the sequel.

3.3. **The study of simple components.** In this Subsection, we study components of $\mathcal{W}^1_{2r-2+n}$ whose generic member (C, A) gives rise to a *simple* Lazarsfeld-Mukai vector bundle $E(C, A)$. The relative dimension of *simple components* dominating $|2H|$ is determined thanks to a more general result due to Pareschi.

Theorem 3.2 ([P95, Theorem 2, p. 196])**.** *Let S be a $K3$ surface, and L a line bundle on S. Let $\mathcal{W}$ be an irreducible component of $\mathcal{W}^r_d(|L|_s)$. Suppose that for a generic pair $(C, A) \in \mathcal{W}$, the line bundle A is base-point-free, $h^0(C, A) = r+1$ and the associated vector bundle $E(C, A)$ is simple. If the Petri map $\mu_{0,A}$ is not injective, then the differential*

$$(d\pi_S)_{|(C,A)} : T_{(C,A)}\mathcal{W} \to T_C|L| = H^0(C, K_C)$$

is not surjective.

By standard Brill-Noether theory [ACGH85] this result immediately yields the following.

Corollary 3.3. *The relative dimension of the simple components of $\mathcal{W}^r_d(|L|_s)$ dominating $|L|$ equals $\rho(p_a(L), r, d)$.*

Notice that in our case, $\rho(4r-3, 1, 2r-2+n)$ is negative for $n \in \{0, 1\}$, it equals 1, for $n = 2$, and it equals 3, for $n = 3$. In particular, there is no dominating simple component in the cases $n = 0$ and $n = 1$.

Theorem 3.2 is stated, and proved, in [P95] under the hypothesis that the linear system $|L|$ does not contain reducible or multiple curves. This condition on $|L|$ is only used in order to insure the simplicity of the associated bundles $E(C, A)$, which we assume. For the convenience of the reader, since we have slightly changed the hypothesis, we sketch below Pareschi's very nice infinitesimal argument.

Proof of Theorem 3.2. Consider the map

$$\mu_{1,A,S} : \ker(\mu_{0,A}) \to H^1(C, \mathcal{O}_C)$$

defined as the composition of the Gaussian map (see [AC81])

$$\mu_{1,A} : \ker(\mu_{0,A}) \to H^0(C, 2K_C),$$

with the transpose $\delta^{\vee}_{C,S}$ of the Kodaira-Spencer map

$$\delta_{C,S} : H^0(C, N_{C/S} = K_C) \to H^1(C, T_C).$$

The first fact is that, by standard first-order deformation theory, (see for instance [CGGH, §2(c)]) one has

$$\text{(3.3.1)} \qquad \operatorname{Im}(d\pi_S)_{|(C,A)} \subset \operatorname{Ann}(\mu_{1,A,S}) \subset H^1(C, \mathcal{O}_C)^* \cong H^0(C, K_C).$$

On the other hand, Pareschi [P95, Lemma 1] shows that, up to a scalar factor, $\mu_{1,A,S}$ coincides with the coboundary map:

$$\text{(3.3.2)} \qquad H^0(C, M_A \otimes K_C(-A)) \to H^1(C, \mathcal{O}_C)$$

of the exact sequence (2.1.6) twisted by $K_C(-A)$:

$$0 \to \mathcal{O}_C \to E(C,A)^* \otimes K_C(-A) \to M_A \otimes K_C(-A) \to 0.$$

Now, by hypothesis, $\ker(\mu_{0,A}) \cong H^0(C, M_A \otimes K_C(-A))$ is not zero. Moreover, again by hypothesis,

$$h^0(S, E(C,A) \otimes E(C,A)^*) = 1.$$

In particular, since twisting (2.1.2) with $E(C,A)^*$ we have that

$$H^0(S, E(C,A) \otimes E(C,A)^*) \cong H^0(C, E(C,A)^* \otimes K_C(-A)),$$

by (3.3.2) we get that $\mu_{1,A,S}$ is injective, hence

$$\mu_{1,A,S} \text{ is not the zero map.}$$

Therefore

$$\operatorname{Ann}(\mu_{1,A,S}) \subsetneq H^1(C, \mathcal{O}_C)^* \cong H^0(C, K_C)$$

and, by (3.3.1), the differential $d\pi_{S|(C,A)}$ cannot be surjective.

□

3.4. **The study of non simple components.** In this section we want to study the irreducible components of $\mathcal{W}^1_{2r}(|2H|_s)$, whose generic member (C,A) has the property that A is a complete base-point-free pencil of degree $2r$, with the associated bundle $E(C,A)$ not simple.

We prove the following.

Theorem 3.4. *Let $\mathcal{W}$ be a non simple irreducible component. If $\mathcal{W}$ dominates $|2H|$, then its relative dimension equals* 1.

A key tool to study non simple $E := E(C,A)$ is provided by Lemma 2.4. Accordingly, we have an exact sequence :

$$\text{(3.4.1)} \qquad 0 \to M \to E \to N \otimes I_\xi \to 0.$$

Since in our case $\operatorname{Pic}(S) = \mathbb{Z}[H] \oplus \mathbb{Z}[\ell]$, the line bundles M and N are of the form

$$M = aH + b\ell, \quad N = a'H + b'\ell.$$

As $c_1(E) = 2H$, we have that $a + a' = 2$, and $b' = -b$, and using the fact that $h^0(M), h^0(N) \geq 2$, we get

$$a = a' = 1.$$

Moreover,

$$c_2(E) = M.N + \lg(\xi) = 2r.$$

In conclusion, two cases may occur :

($\star$) $b = 0$ and $\lg(\xi) = 2$;

($\star\star$) $b = 1,\ b' = -1$ and ξ is empty;

the case $b = -1,\ b' = 1$ is excluded since N is globally generated.

We point out the following useful fact.

Lemma 3.5. *Let $\mathcal{E}$ be a torsion-free sheaf given by a non-trivial extension*

$$0 \to \mathcal{O}_S(H) \to \mathcal{E} \to \mathcal{O}_S(H) \otimes I_\xi \to 0, \tag{3.4.2}$$

where $\xi \subset S$ is a zero-dimensional subscheme with $\lg(\xi) = 2$. Then $\mathcal{E}$ is locally free.

Proof. Suppose $\mathcal{E}$ was not locally free, and denote $E = \mathcal{E}^{**}$. As in [OSS80], cf. [La97, Proof of Proposition 3.9], there exists a subscheme $\eta \subsetneq \xi$, and a commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_S(H) \arrow[r] \arrow[d, equal] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{O}_S(H) \otimes I_\xi \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_S(H) \arrow[r] & E \arrow[r] \arrow[d] & \mathcal{O}_S(H) \otimes I_\eta \arrow[r] \arrow[d] & 0 \\
 & & E/\mathcal{E} \arrow[r, equal] \arrow[d] & I_{\eta \subset \xi} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{3.4.3}$$

where $E/\mathcal{E}$ is supported on the singular locus of $\mathcal{E}$. Since $\lg(\eta) \leq 1$, and E is locally free, we deduce that $\eta = \emptyset$. Then E is split, and therefore the extension (3.4.2) is trivial contradicting the hypothesis. $\square$

Lemma 3.6. *For any non simple bundle E, the extension (3.4.1) is uniquely determined.*

Proof. We observe first that $h^0(S, E(-M)) = 1$. Indeed, in the case ($\star$), we use $h^0(S, I_\xi) = 0$, as $\xi \neq \emptyset$. In the case ($\star\star$), we use $h^0(S, \mathcal{O}_S(-2\ell)) = 0$.

Next, we prove that one E cannot lie in two different extensions:

$$0 \to \mathcal{O}_S(H) \to E \to \mathcal{O}_S(H) \otimes I_\xi \to 0,$$

and

$$0 \to \mathcal{O}_S(H + \ell) \to E \to \mathcal{O}_S(H - \ell) \to 0.$$

If we found two extensions as above, then, using $h^0(S, E(-H)) = 1$ as remarked above, we would be led to a commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_S(H) \arrow[r, hook] \arrow[d, hook] & E \arrow[d, equal] \\
\mathcal{O}_S(H + \ell) \arrow[r, hook] & E
\end{tikzcd} \tag{3.4.4}$$

Furthermore, by obvious reasons, we would obtain an inclusion

$$\mathcal{O}_S(H+\ell)/\mathcal{O}_S(H) \hookrightarrow E/\mathcal{O}_S(H).$$

This is absurd, as $E/\mathcal{O}_S(H) = \mathcal{O}_S(H) \otimes I_\xi$ is torsion-free, whereas $\mathcal{O}_S(H+\ell)/\mathcal{O}_S(H)$ is supported on ℓ. □

Then, since the data of an extension (3.4.1) consists of a (possibly empty) 0-dimensional subscheme ξ of S and an element $t \in \mathbb{P}H^1(S, M^\vee \otimes N \otimes I_\xi)^*$, we obtain:

Proposition 3.7. *The parameter space $\mathcal{P}$ for the non simple vector bundles E is either isomorphic to $S^{[2]}$ in the case $(\star)$, or to a projective plane $\mathbb{P}^2$ in the case $(\star\star)$.*

Proof. In the case $(\star)$, the parameter space in question parametrises pairs (t, ξ) with $\xi \in S^{[2]}$, and $t \in \mathbb{P}H^1(S, M^\vee \otimes N \otimes I_\xi)^* = \mathbb{P}H^1(S, I_\xi)^*$ *arbitrary.* Since $h^1(S, I_\xi) = 1$, the parameter space coincides actually with $S^{[2]}$, *see* also Lemma 3.5. In the case $(\star\star)$, we observe that the split bundle is not of type $E(C, A)$, as the union of the zero loci of two sections: one of $\mathcal{O}_S(H+\ell)$ and the other one of $\mathcal{O}_S(H-\ell)$ is a reducible curve in $|2H|$. □

Remark 3.8. The existence of a universal extension over $S^{[2]}$ is insured by a more general result due to H. Lange (see [L83, Proposition 4.2 and Remark 3.5]).

Now consider the Grassmann bundle $\mathcal{G} \xrightarrow{p} \mathcal{P}$, whose fiber at a point $[E] \in \mathcal{P}$ is the Grassmannian of two-dimensional subspaces of the global sections of E:

$$p^{-1}([E]) = \mathrm{Gr}(2, H^0(S, E)).$$

The dimension of $\mathcal{G}$ is

$$\dim(\mathcal{G}) = \dim(\mathcal{P}) + \dim(\mathrm{Gr}(2, H^0(S, E))) = \dim\mathcal{P} + 4r - 4 \tag{3.4.5}$$

and there is a rational map

$$d : \mathcal{G} \dashrightarrow |2H|, \quad ([E], \Lambda) \mapsto d_E(\Lambda) \tag{3.4.6}$$

where the map d_E is the determinant map, defined in (2.1.3). The utility of $\mathcal{G} \overset{d}{\dashrightarrow} |2H|$ is made clear by the following result.

Proposition 3.9. *The irreducible components $\mathcal{W}$ of $\mathcal{W}^1_{2r}(|2H|_s)$ whose generic member (C, A) is such that A is base-point-free and the associated bundle $E(C, A)$ is not simple, are birational to the two Grassmann bundles $\mathcal{G}_\star$ and $\mathcal{G}_{\star\star}$ corresponding to the cases $(\star)$ and $(\star\star)$. Moreover, denoting these components by $\mathcal{W}_\star$ and $\mathcal{W}_{\star\star}$, and by $f_\star$ and $f_{\star\star}$ the birational maps between them and the Grassmann bundles, we have that $f_\star$ and $f_{\star\star}$ commute with the maps d and π_S on $|2H|$.*

Proof. We write the proof only for the case $(\star)$. The map $f_\star$ from $\mathcal{G}_\star$ to the corresponding irreducible component $\mathcal{W}_\star$ associates to a generic pair $([E], \Lambda)$ in the Grassmann bundle, the element $(C, A) \in \mathcal{W}_\star$, where $C := d_E(\Lambda)$ and

$$A := \mathrm{Im}(\Lambda \otimes \mathcal{O}_C \hookrightarrow E \otimes \mathcal{O}_C).$$

The map $f_\star$ has degree one, since a pair (C, A), with C smooth, and A base-point-free with $h^0(C, A) = 2$, determines a unique element $([E(C, A)], H^0(C, A)^*) \in \mathcal{G}_\star$. The commutativity of the diagram

$$\begin{tikzcd} \mathcal{W}_\star \arrow[dr, "(\pi_S)_{|\mathcal{W}_\star}"'] & \mathcal{G}_\star \arrow[l, dashed, "f_\star"] \arrow[d, dashed, "d_{|\mathcal{G}_\star}"] \\ & {|2H|} \end{tikzcd} \tag{3.4.7}$$

follows immediately from the description of the birational map $f_\star$ we have given. □

In case $(\star\star)$ the dimension of $\mathcal{P}$ equals 2, so we get

$$\dim(\mathcal{G}_{\star\star}) = 4r - 2.$$

Since $\dim(|2H|) = 4r - 3$, if d is dominant, then

$$\dim(d^{-1}(C)) = 1, \tag{3.4.8}$$

implying that the relative dimension of the component $\mathcal{W}_{\star\star}$ equals one.

The case $(\star)$ is slightly different. Notice that in this case, by (3.4.5), we have

$$\dim(\mathcal{G}_\star) = 4r. \tag{3.4.9}$$

So if $d : \mathcal{G}_\star \dashrightarrow |2H|$ were dominant, the varieties of pencils $W^1_{2r}(C)$ of a generic curve $C \in |2H|$ would have dimension equal to 3, and thus they would not satisfy the linear growth conditions. Theorem 3.4 will then be a consequence of the following result.

Lemma 3.10. *The Grassmann bundle $\mathcal{G}_\star$ does not dominate $|2H|$.*

Proof. We use again Pareschi's infinitesimal approach. Suppose that $\mathcal{W}_\star$ dominates $|2H|$. If $(C, A) \in \mathcal{W}_\star$ is a generic pair, then arguing as in the proof of Theorem 3.2 one obtains that $\ker(\mu_{0,A})$, which, by the base-point-free pencil trick is isomorphic to $H^0(C, K_C(-2A))$, is at least two-dimensional.

On the other hand, if $\mathcal{W}_\star$ dominates $|2H|$, then (3.3.1) implies that $\mu_{1,A,S} \equiv 0$, hence we have an exact sequence

$$0 \to H^0(C, \mathcal{O}_C) \to H^0(C, E^*|_C \otimes K_C(-A)) \to H^0(C, K_C(-2A)) \to 0.$$

We would obtain then $h^0(C, E^*|_C \otimes K_C(-A)) \geq 3$.

On the other hand, twisting the exact sequence

$$0 \to \mathcal{O}_S(H) \to E \to \mathcal{O}_S(H) \otimes I_\xi \to 0$$

by E^*, and recalling that by determinant reasons, $E^* \cong E \otimes \mathcal{O}_S(-2H)$, we get

$$0 \to E(-H) \to E \otimes E^* \to E(-H) \otimes I_\xi \to 0,$$

implying $h^0(S, E \otimes E^*) \leq 2$. Furthermore, the exact sequence

$$0 \to \Lambda \otimes \mathcal{O}_S \to E \to K_C(-A) \to 0$$

twisted by E^* together with the relations $h^0(E^*) = h^1(E^*) = 0$, yields to an isomorphism

$$H^0(S, E \otimes E^*) \cong H^0(C, E^*|_C \otimes K_C(-A)),$$

which lead to a contradiction.

Consequently, the map $\mu_{1,A,S}$ is not identically zero and the irreducible component $\mathcal{W}$ of $\mathcal{W}^1_{2r}(|2H|_s)$ cannot be dominant by (3.3.1). □

We pass now to the proof of Theorem 3.4.

Proof of Theorem 3.4. Non simple components of $\mathcal{W}^1_{2r}(|2H|_s)$, are birational to one of the two Grassmann bundles $\mathcal{G}_\star$ and $\mathcal{G}_{\star\star}$, see Proposition 3.9.

The first one cannot dominate $|2H|$, by Lemma 3.10. If the latter one dominates $|2H|$, then its relative dimension equals 1, by (3.4.8), which we wanted to prove. □

3.5. **Conclusion of proofs of Theorem 1.4 and Proposition 1.3.** As we have analysed several cases, we now make the point and put them together to show how they imply Theorem 1.4. Also, we prove that curves in the image of the non-simple component birational to $\mathcal{G}_\star$ violate Statement (T).

Proof of Theorem 1.4. Let C be a generic curve in $|2H|$. Let $W^1_{2r-2+n}(C)$, $n \in \{0,1,2\}$ be the variety of degree $2r-2+n$ pencils on C. For $n=0$, by [CP95] the dimension of $W^1_{2r-2}(C)$ is zero. For $n=1$, by Lemma 3.1, $W^1_{2r-1}(C)$ has only irreducible components W whose member is a pencil A' on C obtained by adding a base point to a $A \in W^1_{2r-2}(C)$ Hence $\dim(W^1_{2r-1}(C)) = 1$. For $n=2$, and W an irreducible component of $W^1_{2r}(C)$, we have several possibilities.

- **(a)** A generic $A \in W$ is base-point-free.
- **(b)** Any $A \in W$ has base-points.

In case **(a)**, also using [ACGH85, Lemma 3.5, p. 182], we obtain two subcases.

- **(a1)** A generic $A \in W$ is such that $h^0(C,A) = 2$ and $E(C,A)$ is simple.
- **(a2)** A generic $A \in W$ is such that $h^0(C,A) = 2$ and $E(C,A)$ is not simple.

In case **(a1)**, the dimension of such a component is given by Corollary 3.3, and equals 1, if $n=2$.

In case **(a2)**, as C is generic, and, by Lemma 3.10, $\mathcal{G}_\star$ does not dominate $|2H|$, we have that W must be birational to the fibre of $\mathcal{G}_{\star\star}$ over C, by Proposition 3.9. So its dimension equals 1, by (3.4.8).

In case **(b)**, the component W is dominated by an irreducible component of $W^1_{2r-1}(C) \times C$, so we are done, using case **(a)**, and $n = 0,1$.

The conclusion is that for $n = 0,1,2$ we have

$$\dim(W^1_{2r-2+n}(C)) = n,$$

for C generic in $|2H|$, and the theorem is proved. □

Proof of Proposition 1.3. For $n=0$, as thanks to [CP95] the gonality of any smooth curve in $|2H|$ is $2r-2$, by [FHL84] we necessarily have

$$\dim(W^1_{2r-2}(C)) \leq 1.$$

For $n=1$, thanks to Accola's Lemma [Ac81], for any smooth curve C in $|2H|$ we have

$$\dim(W^1_{2r-1}(C)) = \dim(W^1_{2r-2}(C)) + 1 \leq 2.$$

For $n=2$, consider the irreducible component of $\mathcal{W}^1_{2r}(|2H|_s)$ which is birational to $\mathcal{G}_\star$ (notice that $\mathcal{G}_\star$ is not empty by the Serre construction). As we have proved in the previous subsection, $\mathcal{G}_\star$ does not dominate $|2H|$. Hence, taking *any* smooth curve C lying in the image of $\mathcal{G}_\star$, and using (3.4.9) and [FHL84] as we have done before, we get $\dim(W^1_{2r}(C)) \geq 4$, and $\dim(W^1_{2r+1}(C)) \geq 5$. By [FHL84], we obtain

that the dimension of $W^1_{2r-1}(C)$ is at least two; in particular, it equals two. It implies $\dim(W^1_{2r-2}(C))=1$, and $\dim(W^1_{2r}(C))=4$. □

4. Green's conjecture for curves in $|2H+\ell|$.

4.1. **Passing from $|2H|$ to $|2H+\ell|$.** We recall first Green's Hyperplane Section Theorem [G84, Theorem (3.b.7)], which reads, in our case

$$K_{p,1}(S,\mathcal{O}_S(2H))\cong K_{p,1}(X,\mathcal{O}_X(2H)),$$

for any connected curve $X\in|2H+\ell|$, and any positive integer p. We apply this result twice, once for a smooth curve, and once again for a curve with two reducible components. Specifically, we obtain $K_{p,1}(X,K_X)\cong K_{p,1}(C+\ell,\omega_{C+\ell})$, where $X\in|2H+\ell|$, and $C\in|2H|$ are smooth curves, and $C.\ell=x+y$. Next, remark that the exact sequence:

$$0\to\mathcal{O}_\ell(-2)\to\mathcal{O}_{C+\ell}\to\mathcal{O}_C\to 0$$

yields, after tensoring with $\mathcal{O}_S(2H+\ell)$, to an isomorphism of vector spaces $H^0(C+\ell,\omega_{C+\ell})\cong H^0(C,K_C(x+y))$. Similarly, we obtain an inclusion $H^0(C+\ell,\omega^{\otimes 2}_{C+\ell})\subset H^0(C,(K_C(x+y))^{\otimes 2})$. By the definition of the Koszul cohomology groups, we get an isomorphism

$$K_{p,1}(C+\ell,\omega_{C+\ell})\cong K_{p,1}(C,K_C(x+y)).$$

The genus of X is $4r-2$, its gonality equals $2r$, and its Clifford index equals $2r-3$, [ELMS89]. Green's conjecture for X predicts

$$K_{2r,1}(X,K_X)=0.$$

It amounts to prove (by what we have said above):

$$K_{2r,1}(C,K_C(x+y))=0.$$

The curve C is of genus $4r-3$, gonality $2r-2$, and Clifford index $2r-4$, ([ELMS89], [CP95], and [GL87]). Note that $h^0(C,K_C(x+y))=4r-2$, and the vanishing $K_{2r,1}(C,K_C(x+y))=0$ is the one predicted by the Green-Lazarsfeld conjecture for the bundle $K_C(x+y)$.

4.2. **Pencils through $x+y$.** We prove the following.

Lemma 4.1. *Let $C\in|2H|$ be any smooth curve, and $x,y\in C$ its intersection points with the line ℓ. For any integer $n\geq 0$, there is no base-point-free line bundle A on C with $h^0(C,A)=2$, $\deg(A)=2r-2+n$, and $h^0(C,A(-x-y))\neq 0$.*

Proof. We argue by contradiction. Suppose there exists complete base-point-free pencil A of degree $(2r-2+n)$ on C such that

$$h^0(C,A(-x-y))\neq 0.$$

Since A is base-point-free, and $h^0(C,A)=2$, we necessarily have

$$h^0(C,A(-x-y))=1.$$

Consider the associated vector bundle $E:=E(C,A)$. Twisting the exact sequence (2.1.2) by $\mathcal{O}_S(\ell)$ we obtain

$$0\to H^0(C,A)^*\otimes\mathcal{O}_S(\ell)\to E\otimes\mathcal{O}_S(\ell)\to K_C(-A+x+y)\to 0. \tag{4.2.1}$$

Using the Riemann-Roch theorem for surfaces, one checks that

$$h^1(S, \mathcal{O}_S(\ell)) = 0.$$

Therefore from (4.2.1) we get

$$h^0(S, E \otimes \mathcal{O}_S(\ell)) = 2 + h^0(K_C(-A + x + y)) = 2 + h^1(A(-x - y)) = 2r + 3 - n,$$

where the last equality also follows from Riemann-Roch.

On the other hand from (2.1.2) one also gets

$$h^0(S, E) = 2 + h^1(C, A) = 2r + 2 - n.$$

Consider the exact sequence defining ℓ, twisted by $E \otimes \mathcal{O}_S(\ell)$:

$$0 \to E \to E \otimes \mathcal{O}_S(\ell) \to E_{|\ell}(-2) \to 0.$$

As $h^1(S, E) = 0$, from the above exact sequence we get

$$h^0(E_{|\ell}(-2)) = h^0(E \otimes \mathcal{O}_S(\ell)) - h^0(E) = 1. \tag{4.2.2}$$

Now

$$E_{|\ell}(-2) = \mathcal{O}_\ell(a) \oplus \mathcal{O}_\ell(b).$$

Since $c_1(E) = 2H$, we have $a + b = 0$. So we can rewrite (4.2.2) :

$$h^0(\ell, \mathcal{O}_\ell(a)) + h^0(\ell, \mathcal{O}_\ell(-a)) = 1$$

which is absurd. □

Remark 4.2. In the case $n = 0$, as the gonality of any smooth curve in $|2H|$ equals $2r-2$, from the previous Lemma we deduce that for any smooth curve C in the linear system $|2H|$, there is no $\mathfrak{g}^1_{2r-2}$ passing through the intersection points of C with the line ℓ.

We arrive which makes crucial use of Theorem 1.4.

Proposition 4.3. *Let C be a generic curve in the linear system $|2H|$, and $\{x_0, y_0\} = C \cap \ell$. For three generic cycles $x_1+y_1, x_2+y_2, x_3+y_3 \in C^{(2)}$, and for any $n \in \{1, 2, 3\}$, there is no line bundle $A \in W^1_{2r-2+n}(C)$, verifying*

$$h^0(C, A(-x_0 - y_0)) \neq 0,$$

and

$$h^0(C, A(-x_{i_j} - y_{i_j})) \neq 0$$

for any set of indices $\{i_1, \dots i_n\} \subset \{1, 2, 3\}$.

Proof. We argue as in [A05] with the difference that we cannot simply invoke the genericity of the pairs of points, since $x_0 + y_0$ is fixed. Nevertheless, we can overpass this particularity using Lemma 4.1.

We have the following.

Claim 4.4. *The incidence variety inside $\prod_n C^{(2)} \times W^1_{2r-2+n}(C)$,*

$$\Xi := \{(x_1 + y_1, \dots, x_n + y_n, A),\ h^0(C, A(-x_i - y_i)) \geq 1,\ \textit{for all } i = 0, \dots, n\}$$

is at most $(2n-1)$-dimensional.

The proof of the claim is done by analysing all the possible irreducible components $\mathcal{W}$ of the universal family $\mathcal{W}^1_{2r-2+n}(|2H|_s)$ dominating $|2H|$, and applying Lemma 4.1. We have two cases according to the behaviour of a generic point $(C,A) \in \mathcal{W}$.

(a) A is base-point-free.

(b) A has base-points.

In case **(a)**, by [ACGH85, Lemma 3.5, p.182] we necessarily have $h^0(C,A)=2$, and hence we may apply Lemma 4.1 to get to a contradiction. In conclusion this case does not occur.

In case **(b)**, let $z \in C$ be a base-point of A. Up to subtracting from A all the base-points different from x_0 and y_0 and applying Lemma 4.1 to get to a contradiction, we may assume $z \in \{x_0, y_0\}$. We consider the incidence variety Ξ' inside $\prod_n C^{(2)} \times W^1_{2r-3+n}(C)$:

$$\Xi' := \{(x_1+y_1, \dots, x_n+y_n, A'),\ h^0(C, A'(-x_i-y_i)) \geq 1, \text{ for all } i=1,\dots,n\}$$

(notice that the difference between Ξ and Ξ' is that for the latter we are not imposing that the line bundles A' pass through x_0+y_0). We have two injective maps from Ξ' to Ξ :

$$j_{x_o} : \Xi' \hookrightarrow \Xi; \quad A' \mapsto A := A' + x_0$$

and

$$j_{y_o} : \Xi' \hookrightarrow \Xi; \quad A' \mapsto A := A' + y_0.$$

Consider the images $j_{x_o}(\Xi')$ and $j_{y_o}(\Xi')$. They are closed inside Ξ, and moreover, thanks to Lemma 4.1, we have

$$\Xi = j_{x_o}(\Xi') \cup j_{y_o}(\Xi').$$

In particular, the dimension of Ξ equals that of Ξ'. Then we may argue as in [A05, p.394] and apply Theorem 1.4 to conclude that Ξ is at most $(2n-1)$-dimensional.

So the claim, and hence the proposition is proved. □

Corollary 4.5. *Let $C \in |2H|$ be a generic curve, and x, y the intersection points with the line ℓ. Then $K_{2r,1}(C, K_C(x+y)) = 0$; in particular, the Green-Lazarsfeld conjecture holds for C.*

Proof. We choose three generic cycles $x_1+y_1, x_2+y_2, x_3+y_3 \in C^{(2)}$, and denote Y be the nodal curve obtained gluing together x with y, and x_i with y_i for all i, and we prove that $K_{2r,1}(Y, \omega_Y) = 0$, arguing similarly to the proof of [A05, Theorem 2]. We reproduce the arguments here for the reader's convenience. Assume by contradiction that $K_{2r,1}(Y, \omega_Y) \neq 0$. Then, by the degenerate version of the Hirschowitz-Ramanan-Voisin result [A05, Proposition 8], there exists a rank one torsion-free sheaf F on Y, with $\chi(F) = 2r-2-g(C)$, and $h^0(F) \geq 2$. This sheaf is either a line bundle or the direct image of a line bundle on a partial desingularization of Y (which cannot be C itself, otherwise we would contradict the fact that $\mathrm{gon}(C) = 2r-2$). Hence $F = \phi_* L$, where $\phi : Z \to Y$ is a partial normalization of $(4-n)$ of the 4 nodes of Y. Let $\psi : C \to Z$ be the normalization of the remaining n-nodes. Then, $\chi(Z, L) = \chi(Y, F) = 2r-2-g$, and $\chi(C, \psi^* L) = 2r-1-g+n$, and the latter implies that $\deg(\psi^* L) = 2r-2+n$. As L is a pencil, for each of the n nodes, there exists a non-zero section vanishing at it. Hence the pencil $\psi^* L$ would contradict Remark 4.2 and Proposition 4.3. So we have proved that $K_{2r,1}(Y, \omega_Y) = 0$.

On the other hand, thanks to [AV03, Lemma 2.3], we have :

$$K_{2r,1}(C, K_C(x+y)) \subset K_{2r,1}(Y, \omega_Y)$$

and the predicted vanishing is proved. The last assertion follows using [A02, Theorem 3]. □

Proof of Theorem 1.5. We have to check that

$$K_{2r,1}(X, K_X) = 0$$

for a smooth curve X in $|2H + \ell|$. As we have shown in §4.1, we have the equality

$$K_{2r,1}(X, K_X) = K_{2r,1}(C, K_C(x+y))$$

where $C \in |2H|$. The latter Koszul cohomology group vanishes by the previous Corollary and the Theorem is proved. □

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